Power spectrum characterization of the Anderson transition

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We examine the power spectrum of the energy level fluctuations of a family of critical power-law random banded matrices whose spectral properties are similar to those of a disordered conductor at the Anderson transition. It is shown analytically and numerically that at the Anderson transition the power spectrum presents $1/f^2$ noise for small frequencies but $1/f$ noise for larger frequencies. The analysis of the region between these two power-law limits gives an accurate estimation of the Thouless energy of the system. Finally we discuss in what circumstances these findings may be relevant in the case of nonrandom Hamiltonians.

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I. INTRODUCTION

The analysis of the level statistics is one of the main tools in the study of quantum complex systems. Part of this interest is due to the fact that, once the model-dependent spectral density is extracted from the spectrum, level correlations of apparently unrelated models shows striking universal features in a variety of physical situations. For instance, in the context of deterministic Hamiltonians, the celebrated Bohigas-Giannoni-Schmit conjecture [1] states the level statistics of a deterministic quantum system whose classical counterpart is fully chaotic depend only on the global symmetries of the system and are described by the prediction of random matrix theory (usually referred to as Wigner-Dyson statistics (WD) [2]). Remarkably the same WD statistics also describes [3] the spectral correlations of a disordered system in the metallic limit. In the strong disorder limit, due to the phenomenon of Anderson localization, eigenvalues are not correlated and the level statistics are universally described by Poisson statistics. For deterministic systems the same statistics are generic of systems whose classical dynamics is integrable [4]. These universal features are unveiled by computing different spectral correlators from the unfolded spectrum. Two popular choices are the level spacing distribution $P(s)$ (the probability of having two eigenvalues at a distance s) for short range correlations and the number variance $\Sigma^2(L) = \langle L^2\rangle - \langle L\rangle^2$ (it measures deviations of the number of eigenvalues in an interval L from its mean value) for long range correlations.

A different spectral characterization of universal features has been recently introduced in the context of quantum chaos [5,6]. In [5,6], the unfolded energy levels are considered as elements of a time series. Specifically they compute the power spectrum

$$S(k) = \left| \frac{1}{\sqrt{N}} \sum_{n=1}^{N} \delta_n \exp\left(\frac{-2\pi i k n}{N}\right) \right|^2, \tag{1}$$

of the signal

$$\delta_n = \sum_{i=1}^{n} (s_i - \bar{s}) = \epsilon_{n+1} - \epsilon_1 - n, \tag{2}$$

where $s_i = \epsilon_{i+1} - \epsilon_i$, ϵ_i is the ith unfolded eigenvalue and N is the length of the series. The signal δ_n is by definition the deviation of the ith nearest neighbor spacing s_i from its mean value $\bar{s}$ which is unity for unfolded eigenvalues. According to [5] certain features of $S(k)$ are universal, they do not depend on the details of the Hamiltonian but only on the type of classical dynamics: $S(k) \sim 1/k$ for chaotic and $S(k) \sim 1/k^2$ for integrable motion.

Signatures of universality are also found in the statistical properties of the eigenfunctions. Thus Poisson statistics is associated with exponential localization and WD statistics is typical of systems in which the eigenstates are delocalized and can be effectively represented by a superposition of plane waves with random phases.

Universal features appear in principle only for small energy scales of the order of the mean level spacing such that an initially localized wave packet has already explored the whole phase space available. In other words, universality is related to a certain ergodic limit of the quantum dynamics. For larger energy differences the system has not yet relaxed to the ergodic limit and deviations from universality are expected. For finite disordered systems this scale is determined by the dimensionless conductance $g = E_c/\Delta$ (E_c, the Thouless energy, is a scale of energy associated with the classical diffusion time through sample and Δ is the mean level spacing). Roughly speaking the numerical value of g corresponds with the number of eigenvalues whose spectral correlations are universally described by WD statistics.

Universal features not related to any ergodic limit (they persist beyond the mean level spacing scale) have also been observed in a disordered system at the metal-insulator transition also referred to as Anderson transition (AT). By universality in this case we mean that certain parameters characterizing the AT as the slope of the number variance, the value of the dimensional conductance, or the set of multifractal dimensions D_q (see below for a definition) do not depend on boundary conditions, system shape, or the microscopic details of the disordered potential. It is by now well established that a disordered system with short range hopping in more than two dimensions undergoes an AT [7,8] at the center of the band for a critical amount of disorder (for critical we mean a disorder such that, if increased, all the states in the band become exponentially localized). The dimensionless conductance g at the AT is about unity so spectral correlations beyond the mean level spacing scale describe truly dynamical features of the system.

Indeed signatures of an AT are found in both level statistics and eigenfunctions. Systems belonging to this new universality class have multifractal eigenstates. Intuitively multifractality means that the eigenstates have structures at all scales. In a more formal way multifractality is defined through the anomalous scaling of the eigenfunctions moments $\mathcal{P}_q = \int d^d r |\psi(\mathbf{r})|^{2q} \propto L^{-D_q(q-1)}$ with respect to the sample size L, where D_q is a set of different exponents describing the AT [9].

Level statistics at the AT (commonly referred to as critical statistics [10]) is intermediate between WD and Poisson statistics. Although a formal definition is still missing, typical features of critical statistics include: scale invariant spectrum [11], level repulsion, and linear number variance [$\Sigma_2(L) \sim \chi L$] [12] as for a insulator ($\chi=1$) but with a slope smaller $\chi<1$ [0.27 for the three-dimensional (3D) Anderson transition]. Similar spectral properties have also been found in random matrix models based on soft confining potentials [13], effective eigenvalue distributions [14,15] related to the Calogero-Sutherland model [16] at finite temperature, and random banded matrices with power-law decay [17]. The last one is specially interesting since an AT (for the case of $1/r$ decay) has been established analytically by mapping the problem onto a nonlinear σ model.

In this paper we propose an alternative spectral characterization of the Anderson transition based on the analysis of the power spectrum $S(k)$ introduced above. We will also show that the study of $S(k)$ provides with an accurate way to locate the Thouless energy of a disordered system. Finally we will discuss the relevance of our findings for nonrandom Hamiltonians. For instance, it will be shown that $S(k) \sim 1/k^2$ is not exclusive of quantum systems whose classical counterpart is integrable. Consequently a precise classificatory scheme based on $S(k)$ must include other features of $S(k)$ in addition to the exponent of the power-law decay.

The organization of the paper is as follows. In Sec. II we introduce the model to be investigated. In Secs. III and IV the power spectrum $S(k)$ is evaluated both analytically and numerically for a broad range of parameters. Based on these findings we propose a spectral characterization of the Anderson transition. Finally in Sec. V we discuss in what situations our findings may be relevant for nonrandom quantum systems.

II. THE MODEL

Unlike WD or Poisson statistics, critical statistics is not parameter free. Together with generic properties such as scale invariance, level repulsion, and linear number variance $\Sigma^2(L)=\chi L$ $L \gg 1$, there are also features as the numerical value of the slope of the number variance χ which, for short range Anderson models, depends on the Euclidean dimension d of the sample. Thus for the lowest critical dimension $d=2+\epsilon$ ($\epsilon>0$) [8], $\chi \sim \epsilon \ll 1$. In the opposite limit $d \gg 1$, the slope $\chi \leqslant 1$ is close to the unity similar to the case of an insulator.

In this paper, instead of looking at short range Anderson models, we will focus on certain generalized random matrix models which have been shown to reproduce critical statistics [10] with great accuracy. An advantage of these models is that exact analytical solutions are available in a certain region of parameters [13,15,17].

We investigate the ensemble of random complex Hermitian matrices $\hat{H}$ whose matrix elements H_{ij} are independently distributed Gaussian variables with zero mean $\langle H_{ij}\rangle=0$ and variance

$$\langle |H_{ij}|^2\rangle = \frac{1}{2}\left[1 + \frac{1}{b^2}\frac{\sin^2[\pi(i-j)/N]}{(\pi/N)^2}\right]^{-1}. \tag{3}$$

For any value of the bandwidth $0<b<\infty$, the spectral correlations are given by critical statistics and the eigenvectors are multifractal exactly as in the conventional AT in $2<d<\infty$ [17]. The limit $b\to\infty$ corresponds with the standard Gaussian unitary ensemble (GUE) of random matrices. The region $b \gg 1$ (weak diagonal disorder, $\chi \ll 1$) corresponds with $d=2+\epsilon$ ($\epsilon \ll 1$) and the $b \ll 1$ limit with $d \gg 1$ and $\chi \leqslant 1$ (strong diagonal disorder). For matrices with unitary symmetry these two limits are accessible to analytical techniques [17,18]. Here we do not discuss the details of these calculations but just enumerate certain results we will use later on in the calculation of the power spectrum $S(k)$.

For $b \gg 1$, level statistics can be rigorously investigated after mapping the random banded matrix onto a supersymmetry σ model. It can be shown [17] that, in this limit, the connected part of two level correlation function (TLCF) is given by

$$R_2(s) = \frac{\langle \rho(s/2)\rho(-s/2)\rangle}{\langle \rho(0)\rangle^2} - 1 = \delta(s) - \frac{1}{16b^2}\frac{\sin^2(\pi s)}{\sinh^2(\pi s/4b)}, \tag{4}$$

where $\rho(s=E/\Delta)=\Sigma_i \delta(s-s_i)$ is the spectral density in units of the mean level spacing $\Delta = 1/\langle \rho(0)\rangle$, brackets stand for ensemble average and $\langle \rho(0)\rangle$ is the mean spectral density in the region of the spectrum to be studied.

For $b \ll 1$ various spectral correlators can also be calculated explicitly by a virial expansion around the Poisson limit [18] (for a more heuristic approach see [17]).

The region $b \sim 1$ is not yet accessible to analytical techniques. However there is a related exactly solvable random matrix model [15] which to leading order has the same TLCF in the two regions ($b \ll 1, b \gg 1$) discussed above. Its joint probability distribution is given by

$$P(H,b) = \int dU e^{-1/2\ \mathrm{Tr} HH^\dagger} e^{-b/2\ \mathrm{Tr}[U,H][U,H]^\dagger}. \tag{5}$$

Here, the $N \times N$ matrices H and U are Hermitian and unitary, respectively, and the integration measure dU is the Haar measure. Despite its complicated form, it can be shown

[14,15] that the joint distribution of eigenvalues of H is equal to the diagonal element of the density matrix of a system of free spinless fermions at finite temperature $T=1/b$ confined in a harmonic potential. By using elementary statistical mechanics techniques it can be shown that the TLCF for arbitrary b is given by [14,15]

$$R_2(s) = \delta(s) - \left(\int_0^\infty \frac{\cos[\pi st/\rho(0)]}{\rho(0)} \frac{1}{1+ze^{t^2}} dt \right)^2, \qquad (6)$$

where $h=1/2\pi b$, $z=1/(e^{1/h}-1)$, and $\rho(0)=\int_0^\infty 1/(1+ze^{t^2})$. We shall use this expression in the analytical evaluation of the power spectrum for intermediate values of b and then check its validity by carrying out numerical simulations of the random banded model Eq. (3).

III. ANALYTICAL EVALUATION OF $S(k)$. POWER SPECTRUM CHARACTERIZATION OF CRITICAL STATISTICS

Our goal is to compute the power spectrum

$$\langle S(k) \rangle = \left\langle \left| \frac{1}{\sqrt{N}} \sum_{n=1}^N \delta_n \exp\left(\frac{-2\pi ikn}{N} \right) \right|^2 \right\rangle, \qquad (7)$$

with $\delta_n = \sum_{i=1}^n s_i - \bar{s} = \epsilon_{n+1} - \epsilon_1 - n$, ϵ_n represents the nth unfolded eigenvalue of the critical random banded model Eq. (3) and brackets stand for ensemble average.

In a first stage we evaluate $\langle S(k) \rangle$ in a continuous approximation, $\langle S(k) \rangle = \langle S(k) \rangle_{cont}$, $\Sigma \to \int$, $s_i \to \rho(\epsilon) d\epsilon$, and $\bar{s} \to \bar{\rho}(\epsilon') d\epsilon'$ where $\rho(\epsilon) = \sum_i \delta(\epsilon - \epsilon_i)$ is the full spectral density and $\bar{\rho}(\epsilon)$ is just the smooth part of it (the one utilized to unfold the spectrum).

The power spectrum is now given by

$$\begin{aligned} \langle S(k) \rangle_{cont} &= \left\langle \left| \int d\epsilon \delta'(\epsilon) \exp(-2\pi i\epsilon k) \right|^2 \right\rangle \\ &= \left\langle \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} d\epsilon' d\epsilon \exp[-2\pi i(\epsilon \right. \\ &\quad \left. - \epsilon')k] \int^{\epsilon'} \int^{\epsilon} \tilde{\rho}(\alpha) \tilde{\rho}(\alpha') d\alpha' d\alpha \right\rangle, \end{aligned} \qquad (8)$$

with $\delta'(\epsilon) = \int^\epsilon d\epsilon' \tilde{\rho}(\epsilon')$ and $\tilde{\rho}(\epsilon) = \rho(\epsilon) - \bar{\rho}(\epsilon)$.

After integrating by parts in ϵ and ϵ' the above expression simplifies to

$$\begin{aligned} \langle S(k) \rangle_{cont} &= \frac{1}{4\pi^2 k^2} \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} d\epsilon' d\epsilon \exp[-2\pi i(\epsilon \\ &\quad - \epsilon')k] R_2(\epsilon, \epsilon'), \end{aligned} \qquad (9)$$

where $R_2(\epsilon, \epsilon')$ is the TLCF defined previously. Since the spectrum is translational invariant (we are far from the edges) $R_2(\epsilon, \epsilon') = R_2(s = \epsilon - \epsilon')$ and

$$\langle S(k) \rangle_{cont} = \frac{1}{4\pi^2 k^2} \int_{-\infty}^{\infty} \exp(-2\pi isk) R_2(s) = \frac{1}{4\pi^2 k^2} K(k), \qquad (10)$$

where $K(k)$, the Fourier transform of the TLCF, is usually referred to as the spectral form factor.

Once we have obtained an explicit expression for the power spectrum in terms of known quantities as $R_2(s)$ we have to go back to the original discrete formulation. This can be easily done by following standard relations between the discrete and the continuous Fourier transform, here we present the final result and refer to [6,20] for additional details

$$S(k) = \frac{K(t)}{4\pi^2 t^2} + \sum_{q=1}^{\infty} \left[\frac{K(t+q)}{4\pi^2 (t+q)^2} + \frac{K(q-t)}{4\pi^2 (q-t)^2} \right] + \beta, \qquad (11)$$

with $t=k/N$ and β a constant given by $\beta = [K(0)-1]^2/12$ which accounts for the differences between the fluctuations of $\int^{\epsilon'} \int^{\epsilon} \langle \tilde{\rho}(\alpha) \tilde{\rho}(\alpha') \rangle d\alpha' d\alpha$ and those of the original discrete correlator δ_n (see [20] for details). For the sake of simplicity we set $\langle S(k) \rangle = S(k)$. Finally a closed expression of the power spectrum $S(k)$ as a function of the band size $b = 1/2\pi h$ is obtained by combining Eqs. (11), (4), and (6).

In the region $b \gg 1$ the spectral form factor can be explicitly evaluated by using the TLCF of Eq. (4) [27]

$$\begin{aligned} K(t) = 1 - \frac{1}{2} \bigg[& (1-t) \coth\left(\frac{2-2t}{h} \right) + (1+t) \coth\left(\frac{2+2t}{h} \right) \\ & - 2t \coth(2t/h) \bigg]. \end{aligned} \qquad (12)$$

We can distinguish two different regions. For $t \ll h = 1/2\pi b$, corresponding to eigenvalues separated a distance much larger than the mean level spacing, $K(t) \sim h/2$ is a constant and $S(k) \sim h/(8\pi^2 t^2)$, similar to the case of Poisson statistics. However for Poisson statistics $K(0)=1$ but in our case $K(0)=h/2$. This is an important difference since Poisson statistics is associated with eigenstates exponentially localized but for $K(0) \neq 1$ the eigenstates are multifractal and level statistics are described by critical statistics. It seems that the exponent of the decay of $S(k)$ does not specify completely the nature of the quantum dynamics. We will go back to this point when we discuss applications of our work to nonrandom Hamiltonians. In passing we mention that the multifractal dimension D_2 may be inferred from the power spectrum $S(k)$. According to Ref. [26], $K(0) = (d-D_2)/2d$ and consequently, for sufficiently small k, $S(k) = [(d-D_2)/2d](1/4\pi^2 t^2)$ where d is the space dimension.

In the opposite limit $t \gg h = 1/2\pi b$, $K(t) = t$, and $S(t) \sim 1/(2\pi^2 t)$ in agreement with the result for WD statistics (GUE). The transition region between the two types of power-law decay ($1/t$ and $1/t^2$) corresponds to the Thouless energy of the system. As usual it separates short range correlations still controlled by WD statistics from long (beyond the Thouless energy) range correlations in which typical fea-

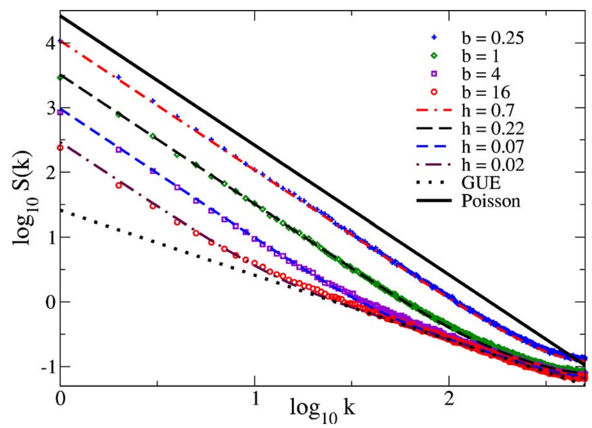

FIG. 1. (Color online) Power spectrum $S(k)$ as a function of k. Symbols represent numerical results (the matrix size is 3000 and the number of eigenvalues considered is $N=1024$ around the center of the band) for the critical random banded model Eq. (3) for different bandwidths b, the power spectrum was evaluated from Eq. (7). Lines represent the analytical prediction of critical statistics Eq. (11) with the TLCF given by Eq. (6). For all bandwidths b the agreement between analytical and numerical results is excellent. For $b \gg 1$ we observe two different regions: $N/k \gg 2\pi b$ corresponding with $S(k) \sim 1/k^2$ and $N/k \ll 2\pi b$ with $S(k) \sim 1/k$ similar to the prediction of WD (GUE). However for $b \ll 1$, $S(k) \sim 1/k^2$ for almost all k.

tures of the AT appear. We have thus found that the power spectrum in the limit $b \gg 1$, corresponding to the case of a disordered system in $2+\epsilon$ dimensions with short range disorder at the AT, has different power-law exponents depending on the spectral region of interest, these differences can be effectively utilized to find signatures of an AT from a given spectrum. We remark that a somehow similar analysis involving only the spectral form factor [not $S(k)$] was first reported in Ref. [27].

Analogously, in the region $b=1/2\pi h \ll 1$, which corresponds to the case of a disordered conductor in $d \gg 1$, a straightforward calculation shows that $K(t)=1-1/\sqrt{2}h$ for $t \ll h$ and then goes to $K(t)=1$ for $t \gg h$. Consequently $S(t) \sim 1/t^2$ up to scales smaller than the mean level spacing. Strictly speaking there is a narrow region for $t>1$ in which $K(t)$ is linear. However the transition region can not be interpreted as a Thouless energy since level statistics deviate from the WD prediction even for distances smaller than the mean level spacing.

As is mentioned previously, the TLCF for intermediate value of b is not yet accessible to analytical treatment. However we shall see in Sec. IV that the conjecture Eq. (6) describes accurately the numerical results (see Fig. 1).

Finally we point out that generic features of $S(k)$ as different regimes of power-law decay with specific exponents are not restricted to the critical random banded model studied in this paper, but should be considered as genuine spectral signatures of an AT, no matter what the microscopic Hamiltonian is. Thus the analysis of just $S(k)$ could be used as a standard technique to detect the transition to localization in a given system. We consider that this spectral characterization of the AT is the most relevant result of this paper.

IV. NUMERICAL RESULTS

We now test the analytical predictions for $S(k)$ of the previous section by numerical diagonalization of critical random banded model, defined by Eq. (3), for different matrix sizes N (almost all of our plots are for $N=3000$ though we tried higher volumes, up to $N=5000$, in order to check that our results are not size dependent). The number of different realizations of disorder is chosen such that for each matrix size the total number of eigenvalues be at least 2×10^6. The eigenvalues thus obtained are unfolded with respect to the mean spectral density. The power spectrum is calculated from Eq. (7) by using a standard fast Fourier transformation routine. Typically up to 35% of the eigenvalues around the center of the band are utilized. We have decided to go beyond the standard recipe of taking no more than 10% of the eigenvalue around the band center in order to investigate to what extent numerical results deviate from the analytical prediction and also to find out whether the main expected features $S(k)$ are robust against finite volume effects (see below for more details about this issue).

In Fig. 1 we have plotted $\log_{10} S(k)$ for different bandwidths b. In all cases the matrix size was 3000 and $S(k)$ was evaluated within a band around the center of the spectrum containing 1024 eigenvalues. As is observed the agreement between analytical [with $R_2(s)$ given by Eq. (6) and $h=1/2\pi b$] and numerical results is excellent for all b's. In agreement also with the analytical prediction we observe that, for $b \gg 1$, the power spectrum switches from $S(k) \propto 1/k$ for $N/k \ll h=1/2\pi b$ to $S(k) \propto 1/k^2$ in the opposite limit. However for $b \ll 1$, $S(k) \propto 1/k^2$ for almost all accessible frequencies. Based on the analytical and numerical results above, we conclude that the AT in a disordered conductor can be satisfactorily detected and examined by looking at the power spectrum $S(k)$ of a signal δ_n given by the fluctuations of the nearest neighbor spacings $s_i=\epsilon_{i+1}-\epsilon_i$ around its mean value $\bar{s}$.

We have also found that $S(k)$ provides with an accurate method to locate the Thouless energy of a generic disordered conductor. As is mentioned previously, the Thouless energy E_c is a scale of energy related with the classical diffusion time through the sample. From a practical point of view the evaluation of $g=E_c/\Delta$ (the Thouless energy in units of the mean level spacing Δ) from a given spectrum is a hard task since it may depend on what spectral correlator is used. Thus $\Sigma^2(L)$ gives a prediction of g bigger than that of $P(s)$ but smaller than that of the spectral rigidity $\Delta_3(L)$ (see [2] for a definition). Another problem is that even for each particular correlator the value of g is somewhat ambiguous since it is far from clear how to locate even approximately the point in which WD statistics cease to be applicable.

Below we show that $S(k)$ provides with a more efficient and precise way to locate g. The idea (see Figs. 2 and 3 is to plot $S(k)$ as a function of N/k. Then, in the region $N/k \gg 2\pi b$, we find the best fit of $S(k)$ to a linear (in a ln scale) curve $S(k) \sim 1/k^{\alpha}$ (from the previous analysis $\alpha \sim 2$). In the opposite limit $S(k)$ should be given by the prediction of WD statistics. We define the Thouless energy as the intersection between the WD prediction and the linear fit.

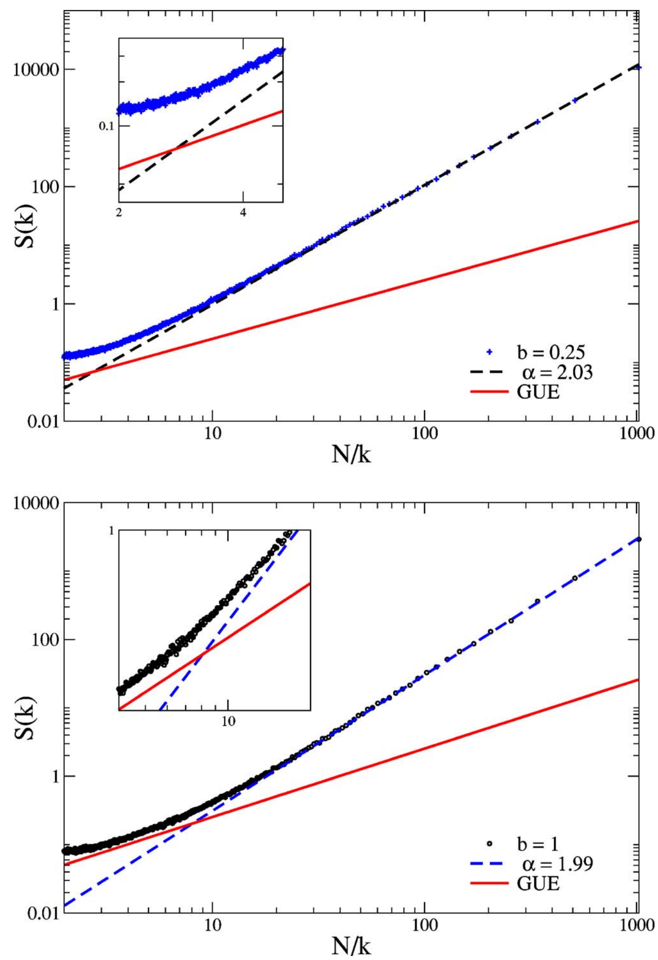

FIG. 2. (Color online) Power spectrum $S(k)$ versus N/k in the limit $b \leqslant 1$. Crosses and circles represent the numerical results (the matrix size is 3000 and the number of eigenvalues considered is $N=1024$ around the center of the band) for the critical random banded model Eq. (3) with $b=0.25, 1$, respectively. The power spectrum $S(k)$ was obtained from Eq. (7). The dashed line corresponds to the best fit $S(k) \propto 1/k^{\alpha}$ (the statistical error of α is about $\Delta\alpha = \pm 0.02$) in the limit $N/k \gg 2\pi b$ and the solid line is the prediction of WD statistics obtained from the GUE. The intersection between the linear fit and the WD statistics prediction corresponds in principle with the dimensionless conductance g (the Thouless energy in units of the mean level spacing) of the system. However we observe that, for $b \leqslant 1$, $S(k)$ is always different from the WD result and consequently no Thouless energy can be defined.

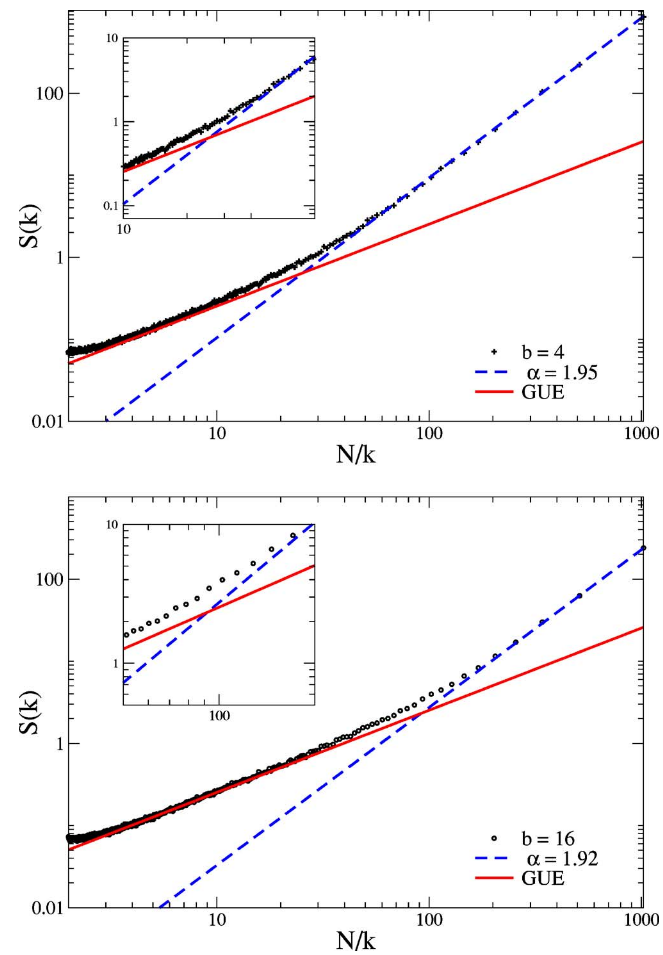

FIG. 3. (Color online) Power spectrum $S(k)$ versus N/k in the limit $b \gg 1$. Crosses and circles represent the numerical results (the matrix size is 3000 and the number of eigenvalues considered is $N=1024$ around the center of the band) for the critical random banded model Eq. (3) with $b=4, 16$, respectively. The power spectrum $S(k)$ was evaluated from Eq. (7). The dashed line corresponds to the best fit $S(k) \propto 1/k^{\alpha}$ (the statistical error of α is about $\Delta\alpha = \pm 0.02$) in the limit $N/k \gg 2\pi b$ and the solid line is the prediction of WD statistics obtained from the GUE. The intersection between the linear fit and the WD prediction corresponds with the Thouless energy, E_c, of the system. Units have been chosen such that the intersection point yields the dimensionless conductance of the system $g=E_c/\Delta$. Thus $g \sim 25$ for $b=4$ and $g \sim 100$ for $b=16$.

In Fig. 2 we see that for $b \gg 1$ these two curves meet at $g \sim 2\pi b \gg 1$, in good agreement with the theoretical prediction. However in the region $b \leqslant 1$ (see Fig. 3), though formally a Thouless energy can be defined through the intersection of the two curves, its interpretation as the limit of applicability of WD statistics is dubious since even in the limit $N/k \ll 2\pi b$ deviations with respect to the WD prediction are clearly visible.

Finally we mention that, as observed in Fig. 1, the best fit of the numerical results does not occur at the analytical estimation $h=1/2\pi b$. There are two reasons for that disagreement: The analytical results are strictly valid only at the center of the band. Eigenstates beyond this region are still critical but are described by an effective bandwidth [19] smaller than b. On the other hand finite size effects may be important in the limit $t=k/N \ll 1$ since we are testing the largest eigenvalue separations. However we have decided not to reduce the spectral window in order to give a full global picture of the power spectrum at the AT. After all, as shown in Fig. 1, the two power law exponents signaling the AT are not affected by these limitations and a very good qualitative agreement with the numerical results is achieved by choosing an effective bandwidth b.

V. APPLICATION TO QUANTUM CHAOS

In this final section we investigate possible applications of our previous findings in the context of quantum chaos and also argue that, contrary to the claims of [5], $S(k) \sim 1/k^2$ is not exclusive of quantum deterministic systems whose classical counterpart is integrable.

Critical statistics and multifractal wave functions are very much universal so they should also appear in deterministic quantum systems. Indeed, in a recent paper [21], we have established a relation between the presence of anomalous diffusion in the classical dynamics, the singularities of a classically chaotic potential and the power-law localization of the quantum eigenstates. Specifically, it was found that for a kicked rotor with Hamiltonian

$$\mathcal{H} = \frac{p^2}{2} + V(q)\sum_n \delta(t - nT) \tag{13}$$

(with $q \in [-1,1)$) both level statistics and eigenfunctions are similar to the ones at the AT (critical statistics) provided that $V(q)$ has a ln (in the simplest case $V(q)=\epsilon_0 \ln|x|$) or a step-like [22] singularity. It was also found in [21] that these findings are universal in the sense that neither the classical nor the quantum properties depend on the details of the potential, but only on the type of singularity. Deviations from the WD statistics not coming from a mixed phase space have also been reported in a variety of systems: a Coulomb billiard [23], the anisotropic Kepler problem [24], a kicked rotor in a well potential [25], and certain pseudointegrable billiards [28,29]. In the last case it was found [28] that level statistics are accurately described by the classical Dyson gas with the logarithmic pairwise interaction restricted to a finite number k of nearest neighbors. Analytical solutions are available for general k. For $k=2$, usually referred to as semi-Poisson (SP) statistics, $R_2(s)=1-e^{-4s}$, $P(s)=4se^{-2s}$, and $\Sigma^2(L)=L/2+(1-e^{-4L})/8$. We have also found [22] that SP statistics describes accurately the spectral correlations of the above kicked rotor with a step-like singularity and also provide with a reasonable description for the case of a logarithmic singularity but only in the region $\epsilon_0 \sim 0.2$.

Due to the simplicity of the TLCF in SP statistics one can evaluate the power spectrum exactly

$$S(k) = \frac{1}{4\pi^2 t^2}\left[1 - \frac{8}{16 + 4\pi^2 t^2}\right], \tag{14}$$

with $t=k/N$. Thus the power spectrum associated to SP statistics presents $1/t^2$ decay in the $t \ll 1$ limit even though the classical dynamics is not integrable, this is also in agreement with the prediction of critical statistics for $b \ll 1$. This explicitly shows $S(k) \sim 1/t^2$ is not always a signature of classical integrable dynamics. Although generically the power spectrum associated with classically integrable systems has this feature, other types of nonintegrable dynamics may have a $1/t^2$ tail as well. In order to fully characterize the classical dynamics from $S(k)$ one has to specify not only the exponent but also an additional point of the curve, for instance $K(0)$. Thus a $S(k) \sim 1/t^2$ behavior only tell us that the form factor is constant. However, as is mentioned previously, the physical properties of the system are strongly modified by a spectral form factor different from unity. In passing we mention that the analytical results for SP statistics are in disagreement with a recent numerical calculation [30].

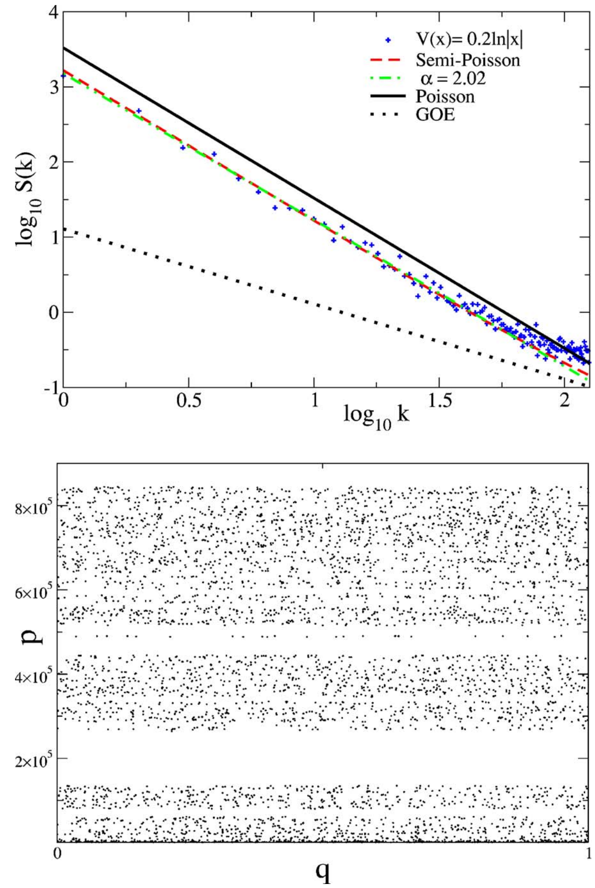

FIG. 4. (Color online) (Left) Power spectrum $S(k)$ obtained from 25 sets of 256 eigenvalues from the numerical diagonalization of the evolution matrix associated to Eq. (13) with $V(q)=0.2\ln|q|$ (cross). The dashed line is the prediction of SP statistics, the dotted-dashed line is the best fit $S(k) \sim 1/k^{\alpha}$ and the dotted line is the prediction of WD statistics as is obtained from the Gaussian orthogonal ensemble (GOE) of random matrices. (Right) Poincare section from a single initial condition $p_0=0.2$ and $q_0=0.6$ after 300 000 iterations.

As a further corroboration of our claims we have evaluated numerically $S(k)$ for the Hamiltonian of Eq. (13) with a potential $V(x)=0.2\ln|x|$. We have diagonalized numerically the evolution matrix associated to the Hamiltonian Eq. (13) for $N=5200$, $S(k)$ is obtained from Eq. (7). In order to improve statistics we divided the original spectrum in 20 sets of 256 eigenvalues. As is shown in Fig. 4, $S(k) \sim 1/t^2$ for almost all $t's$ in close agreement with the prediction of SP or critical statistics. However (see right plot) the associated Poincare section obtained from just a single initial condition is very different from the one corresponding to a classically integrable system.

VI. CONCLUSIONS

We have shown that the power spectrum of the energy level fluctuations at the Anderson transition is characterized by a power spectrum with $1/f^2$ noise for small frequencies and $1/f$ noise for larger frequencies. In the weak disorder limit, the analysis of the transition region between these two power-law limits provides with an accurate estimation of the Thouless energy of the system. As disorder increases the Thouless energy looses its meaning and the power spectrum presents $1/f^2$ noise up to frequencies related to the Heisenberg time of the system. Finally we discuss under what circumstances these findings may be relevant in the context of nonrandom Hamiltonians. Specifically it is shown that the exponent of the power-law decay of $S(k)$ does not fully specify the type of motion of the classical counterpart.

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[1] O. Bohigas, M. J. Giannoni, and C. Schmit, Phys. Rev. Lett. **52**, 1 (1984).

[2] M. L. Mehta, *Random Matrices*, 2nd edition (Academic Press, San Diego, 1991).

[3] K. B. Efetov, Adv. Phys. **32**, 53 (1983).

[4] M. V. Berry and M. Tabor, Proc. R. Soc. London, Ser. A **356**, 375 (1977).

[5] A. Relaño, J. M. G. Gómez, R. A. Molina, J. Retamosa, and E. Faleiro, Phys. Rev. Lett. **89**, 244102 (2002).

[6] E. Faleiro, J. M. G. Gómez, R. A. Molina, L. Muñoz, A. Relaño, and J. Retamosa, Phys. Rev. Lett. **93**, 244101 (2004).

[7] P. W. Anderson, Phys. Rev. **109**, 1492 (1958).

[8] F. Wegner, Z. Phys. B **36**, 209 (1980).

[9] H. Aoki, J. Phys. C **16C**, L205 (1983).

[10] V. E. Kravtsov and K. A. Muttalib, Phys. Rev. Lett. **79**, 1913 (1997); S. M. Nishigaki, Phys. Rev. E **59**, 2853 (1999).

[11] B. I. Shklovskii, B. Shapiro, B. R. Sears, P. Lambrianides, and H. B. Shore, Phys. Rev. B **47**, 11487 (1993).

[12] B. L. Altshuler, I. K. Zharekeshev, S. A. Kotochigova, and B. I. Shklovskii, Sov. Phys. JETP **67**, 62 (1988).

[13] K. A. Muttalib, Y. Chen, M. E. H. Ismail, and V. N. Nicopoulos, Phys. Rev. Lett. **71**, 471 (1993); Y. Chen and K. A. Muttalib, J. Phys.: Condens. Matter **6**, L293 (1994).

[14] A. M. Garcia-Garcia and J. J. M. Verbaarschot, Phys. Rev. E **67**, 046104 (2003); V. E. Kravtsov and A. M. Tsvelik, Phys. Rev. B **62**, 9888 (2000).

[15] M. Moshe, H. Neuberger, and B. Shapiro, Phys. Rev. Lett. **73**, 1497 (1994).

[16] F. Calogero, J. Math. Phys. **10**, 2191 (1969); *ibid.* **10**, 2197 (1969); J. Math. Phys. **12**, 419 (1971).

[17] A. D. Mirlin, Y. V. Fyodorov, F.-M. Dittes, J. Quezada, and T. H. Seligman, Phys. Rev. E **54**, 3221 (1996); E. Cuevas, M. Ortuno, V. Gasparian, and A. Perez-Garrido, Phys. Rev. Lett. **88**, 016401 (2001); F. Evers and A. D. Mirlin, Phys. Rev. Lett. **84**, 3690 (2000); I. Varga, Phys. Rev. B **66**, 094201 (2002); E. Cuevas, Phys. Rev. B **68**, 184206 (2003).

[18] O. Yevtushenko and V. Kravtsov, J. Phys. A **36**, 8265 (2003); V. E. Kravtsov, O. Yevtushenko, and E. Cuevas, e-print cond-mat/0510378.

[19] C. J. Paley, S. N. Taraskin, and S. R. Elliott, Phys. Rev. B **72**, 033105 (2005).

[20] J. B. French, P. A. Mello, and A. Pandey, Ann. Phys. (N.Y.) **113**, 277 (1978); O. Bohigas, P. Leboeuf, and M.-J. Sanchez, Found. Phys. **31**, 489 (2001).

[21] A. M. Garcia-Garcia and J. Wang, Phys. Rev. Lett. **94**, 244102 (2005).

[22] A. M. Garcia-Garcia, Phys. Rev. E **72**, 066210 (2005).

[23] B. L. Altshuler and L. S. Levitov, Phys. Rep. **288**, 487 (1997).

[24] D. Wintgen and H. Marxer, Phys. Rev. Lett. **60**, 971 (1988).

[25] B. Hu, B. Li, J. Liu, and Y. Gu, Phys. Rev. Lett. **82**, 4224 (1999); J. Liu, W. T. Cheng, and C. G. Cheng, Commun. Theor. Phys. **33**, 15 (2000).

[26] J. T. Chalker, V. E. Kravtsov, and I. V. Lerner, JETP Lett. **64**, 386 (1996).

[27] C. Blecken, Y. Chen, and K. A. Muttalib, J. Phys. A **27**, L563 (1994).

[28] E. B. Bogomolny, U. Gerland, and C. Schmit, Phys. Rev. E **59**, R1315 (1999).

[29] E. B. Bogomolny and C. Schmit, Phys. Rev. Lett. **92**, 244102 (2004); O. Giraud, J. Marklof, and S. O'Keefe, J. Phys. A **37**, L303 (2004).

[30] S. N. Evangelou and D. E. Katsanos, Phys. Lett. A **334**, 331 (2005).